Stability, complexity and diversity in random replicator models of ecology and evolutionary game theory

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Game theory

- 1944: von Neumann and Morgenstern
  'Theory of games and economic behaviour'
- 1950s: John Nash, equilibrium concepts
- Nash equilibria seen as only viable outcomes of careful reasoning of rational players
- 1982: John Maynard Smith: 'Evolution and the theory of games', dynamics of a population of irrational players
- Nobel prizes:
  1994: J.C. Harsanyi, John Nash and R. Selten
  2005: R. Aumann, Th. Schelling

A game

- is played by a (finite) number of players $x, \; y, \; z, \ldots$
- each of them has a set of strategies $X, \; Y, \; Z, \ldots$
- and each is paid a payoff depending on his choice of strategy and on the choice of the other players
- different players might have different strategy sets

symmetric versus asymmetric games

Matrix games

E.g. prisoners dilemma

payoff for player 1	2 co-operates	2 defects
1 co-operates	4	0
1 defects	5	3

payoff for player 2	2 co-operates	2 defects
1 co-operates	4	5
1 defects	0	3

Matrix games

Another example: rock-scissors-paper game

rock > scissors, scissors > paper but paper > rock

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Matrix games

- these were all so-called symmetric games: only one type of player

- now an asymmetric game

Battle of the sexes:

- strategies for male: run or stay
- strategies for female: coy or fast

- successful raising of offspring: payoff G for each
- parental investment $-C$ shared if male stays, otherwise borne entirely by female
- long engagement: cost $-E$ for both

Battle of the sexes

- successful raising of offspring: payoff G for each
- parental investment $-C$ shared if male stays, otherwise borne entirely by female
- long engagement: cost $-E$ for both

payoff for male	female coy	female fast
male runs	0	G
male stays	$G - \frac{C}{2} - E$	$G - \frac{C}{2}$

payoff for female	male runs	male stays
female coy	0	$G - \frac{C}{2} - E$
female fast	$G - C$	$G - \frac{C}{2}$

Pure strategies

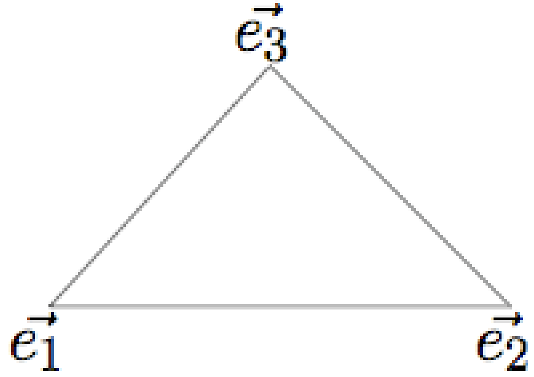
Assume player X has the choice between N pure strategies, labelled by

$$\vec{e_i}^x, i = 1, \ldots, N$$

Then a mixed strategy corresponds to a vector

$$\vec{x} = (x_1, \ldots, x_N), \quad \sum_i x_i = 1$$

x_i is the probability to play pure strategy $\vec{e_i}^x$.



Mixed strategies

- in general will have payoff matrices a_{ij} and b_{ij}

- if player X plays mixed strategy $\vec{x}$ and Y plays $\vec{y}$ then

$$\begin{aligned} \nu^x(\vec{x},\vec{y}) &= \sum_{ij} x_i a_{ij} y_j \\ \nu^y(\vec{x},\vec{y}) &= \sum_{ij} x_i b_{ij} y_j \end{aligned}$$

Nash Equilibria

A Nash equilibrium is a point $(\vec{x}^{\,\star}, \vec{y}^{\,\star})$ such that no player has an incentive to change strategies unilaterally given the other player's choice of strategy:

- $\vec{x}^{\,\star}$ is the best choice for X given Y plays $\vec{y}^{\,\star}$
- $\vec{y}^{\,\star}$ is the best choice for Y given X plays $\vec{x}^{\,\star}$

$$\nu^x(\vec{x}^{\,\star}, \vec{y}^{\,\star}) = \max_{\vec{x}} \nu^x(\vec{x}, \vec{y}^{\,\star})$$

$$\nu^y(\vec{x}^{\,\star}, \vec{y}^{\,\star}) = \max_{\vec{y}} \nu^y(\vec{x}^{\,\star}, \vec{y})$$

Replicator dynamics

replicator equations

$$\frac{d}{dt}x_i(t) = x_i(t)[f_i[x(t)] - f(t)]$$

- evolutionary game theory
- learning dynamics, e.g. acquisition of grammar
- chemical reactions
- interacting species, eco-systems

Replicator dynamics

- differential equations on a simplex S
- population divided into N types $i = 1, \ldots, N$ with proportions x_i
- fitness of i: $f_i(t) = f_i[x_1(t), \ldots, x_N(t)]$

The associated replicator equation reads

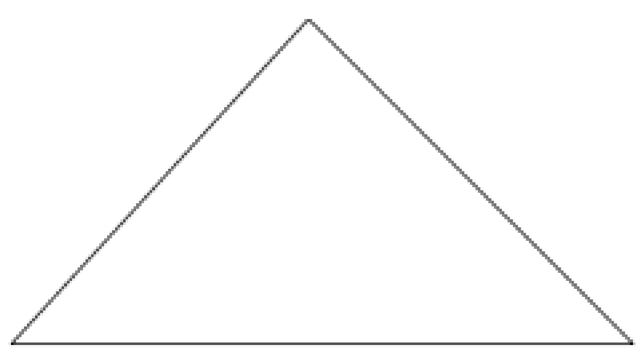
$$\frac{\dot{x}_i(t)}{x_i(t)} = f_i(t) - \overline{f}(t)$$

with $\overline{f}(t) = \sum_i x_i(t) f_i(t)$ the mean fitness

Replicator dynamics

$$\dot{x}_i(t) = x_i(t)\left(f_i(t) - \overline{f}(t)\right)$$

- species fitter than the average prosper
- species less fit than the average decrease in concentration
- $\sum_i x_i = 1$ conserved in time



One population models

- only one type of players X
- e.g in the prisoner's dilemma or rock-scissors-paper game
- symmetric games

$$\dot{x}_i(t) = x_i(t)\left(f_i[x_1(t), \ldots, x_N(t)) - \overline{f}(t)\right)$$

Multi-population models

- multiple types of players $X,\ Y,\ Z,\ldots$ taking different positons in the game
- e.g. male-female in battle of sexes, buyers-sellers in an economy
- asymmetric games

$$
\begin{aligned}
\dot{x}_i(t) &= x_i(t)\left(f_i^x[y_1(t),\ldots,y_N(t)] - \overline{f^x}(t)\right) \\
\dot{y}_j(t) &= y_j(t)\left(f_j^y[x_1(t),\ldots,x_M(t)] - \overline{f^y}(t)\right)
\end{aligned}
$$

Replicator dynamics

Fixed points:

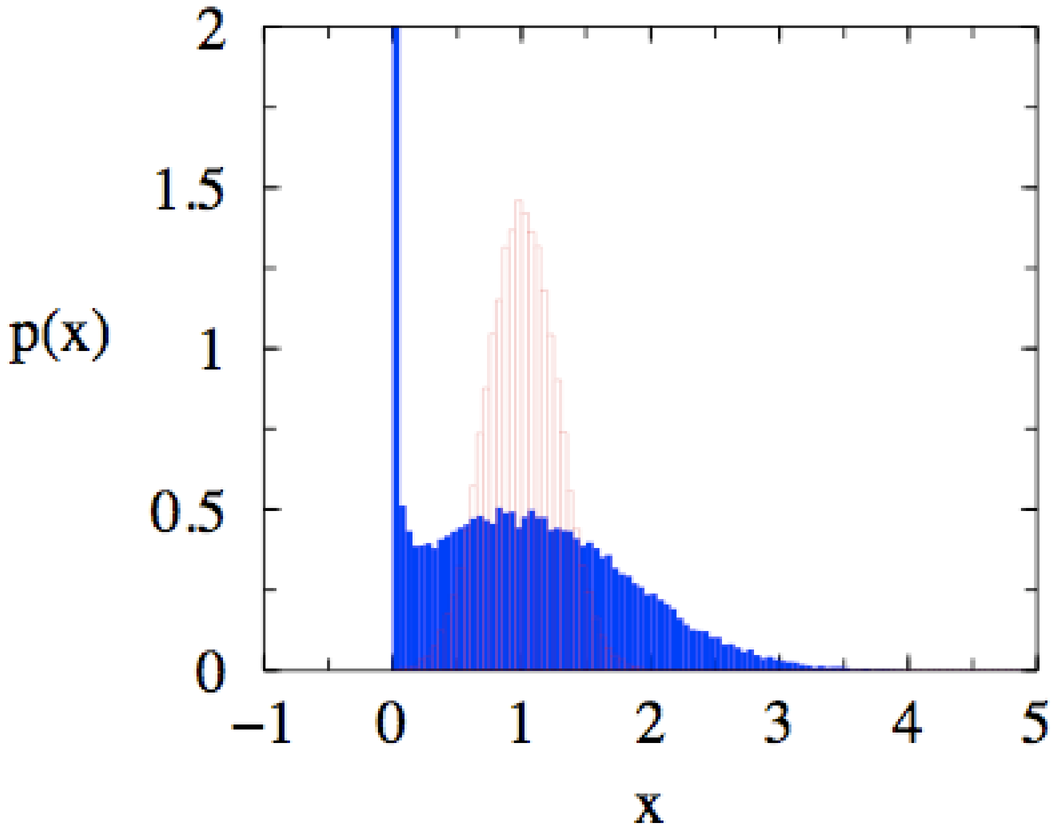
$$0 = x_i \left(f_i - \overline{f} \right)$$

It turns out

- stable fixed points are Nash equilibria
- but Nash equilibria are not necessarily stable FP

Fixed point distribution

- fraction ϕ of surviving species, $x_i > 0$
- distribution $\sim$ truncated Gaussian+$(1-\phi)\delta(x)$



Replicator dynamics

$$\dot{x}_i(t) = x_i(t)\left(f_i[x_1(t), \ldots, x_N(t)) - \overline{f}(t)\right)$$

- games between two players

$$f_i[x_1, \ldots, x_N] = \sum_j J_{ij} x_j$$

- games between p-players

$$f_i[x_1, \ldots, x_N] = \sum_{i_1, \ldots, i_{p-1}} J^i_{i_1, i_2, \ldots, i_{p-1}} x_{i_1} x_{i_2} \cdots x_{i_{p-1}}$$

Replicator dynamics

study "all" matrix games ➡ random payoff matrices

$$\dot{x}_i(t) = x_i(t)\left(\sum_j J_{ij}x_j(t) - \overline{f}(t)\right)$$

vith

- J_{ij} Gaussian couplings, $\overline{J_{ij}^2} = 1/N$
- symmetry of couplings $\overline{J_{ij}J_{ji}} = \Gamma/N$
- diagonal elements $J_{ii} = -2u$
- u denotes 'co-operation pressure', drives the system into the simplex

$$\sum_i x_i(t) = N \qquad \forall t$$

[Opper et al]

Co-operation pressure

[Peschel, Mende, The Prey-Predator model, Springer, 1985]

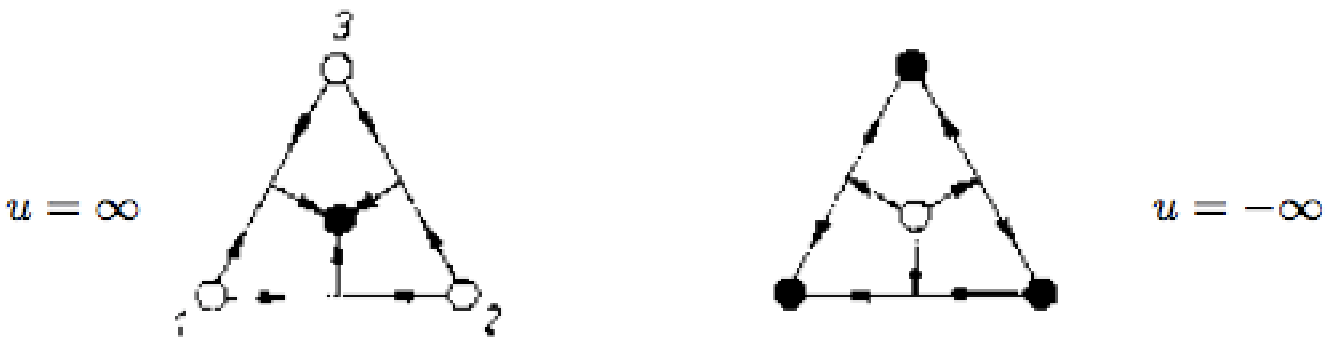


Fig. 89 The influence of λ upon source and sink behaviour

Literature

- statistical mechanics of large one-population systems with random couplings
  - Opper/Diederich PRA ’89, PRL ’92
  - Fontanari, De Oliveira PRL ’00, PRE ’01, PRL ’02, EPJB ’03, PRE ’04 (all replica)
  - Biscari/Parisi J.Phys. A ’95 (1RSB)

- bi-matrix games
  - Berg/Engel PRL ’99
  - Berg/Weigt Europhys. Lett ’99
  - Berg PRE ’00

Random replicator equations

$$\dot{x}_i(t) = x_i(t)\left(\sum_j J_{ij}x_j(t) - \overline{f}(t)\right)$$

with random Gaussian couplings

★ can be solved with techniques from spin glass physics in the thermodynamic limit $N \to \infty$

★ path-integrals, dynamical generating functionals

★ result: stochastic process for a representative strategy/species

★fixed point ansatz gives closed equations for persistent OP

[Opper et al]

Generating functional analysis

Study this with generating functionals.

- advantange over replica:
  - no Lyapunov function required
  - so that GFA can be used also for asymmetric couplings
  - replica theory only for symmetric couplings
- closed laws for dynamical order parameters:
  - correlation function $C(t,t') = N^{-1}\sum_i \overline{\langle x_i(t)x_i(t')\rangle}$
  - response function $G(t,t') = N^{-1}\sum_i \overline{\left\langle \frac{\partial x_i(t)}{\partial h(t')} \right\rangle}$
  - Lagrange multiplier $\overline{f}(t)$

Generating functional analysis

- effective species process

$$\dot{x}(t) = x(t)\left(-2ux(t) - \Gamma\frac{p(p-1)}{2}\int_{t_0}^{t} dt' G(t,t')C(t,t')^{p-2}x(t') + \eta(t) - \overline{f}(t) + h(t)\right)$$

- self-consistent problem

$$C(t,t') = \langle x(t)x(t')\rangle_\star, \quad G(t,t') = \left\langle \frac{\partial x(t)}{\partial h(t')}\right\rangle_\star, \quad \langle x(t)\rangle_\star = 1$$

- $\{\eta(t)\}$ is coloured noise with correlator

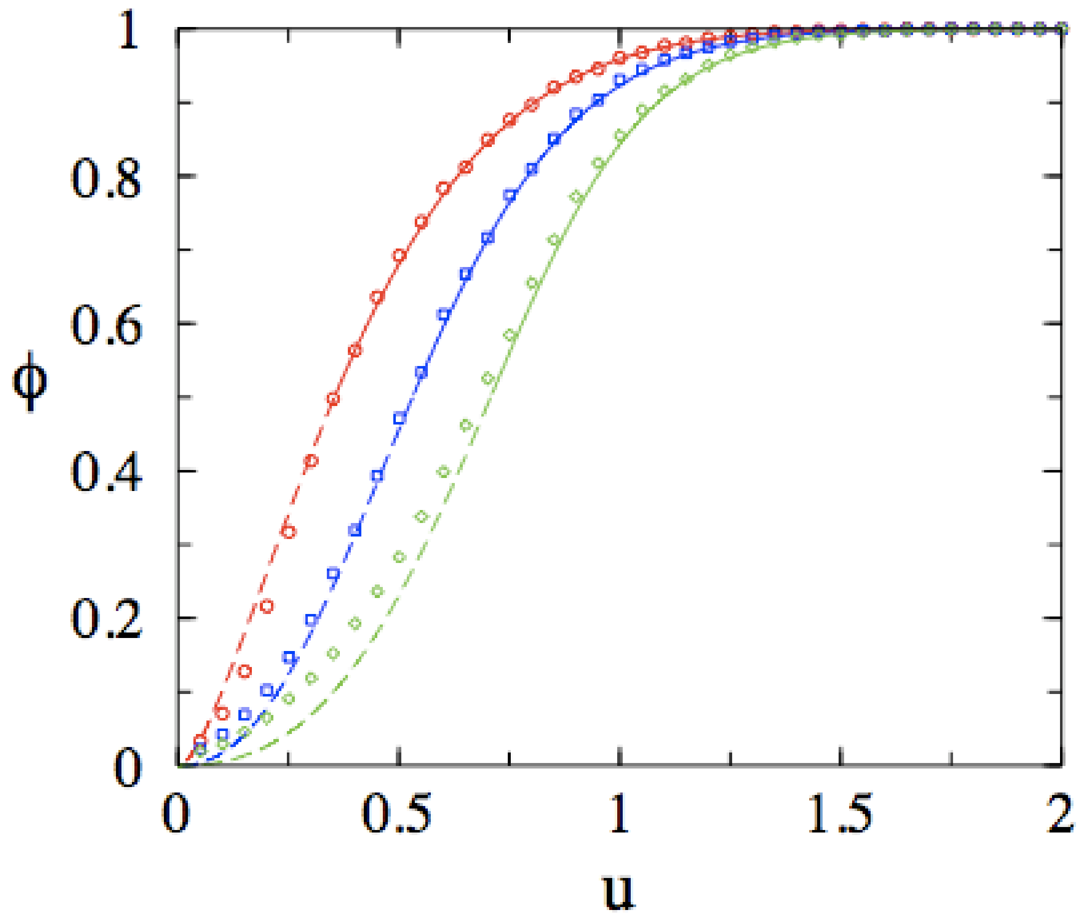
$$\langle \eta(t)\eta(t')\rangle_\star = \frac{p}{2}C(t,t')^{p-1}$$

- retarded interaction $\int_{t_0}^{t} dt' G(t,t') C^{p-2}(t,t')x(t')$

Some results

$(\Gamma = 0, \frac{1}{2}, 1)$

fraction of survivors

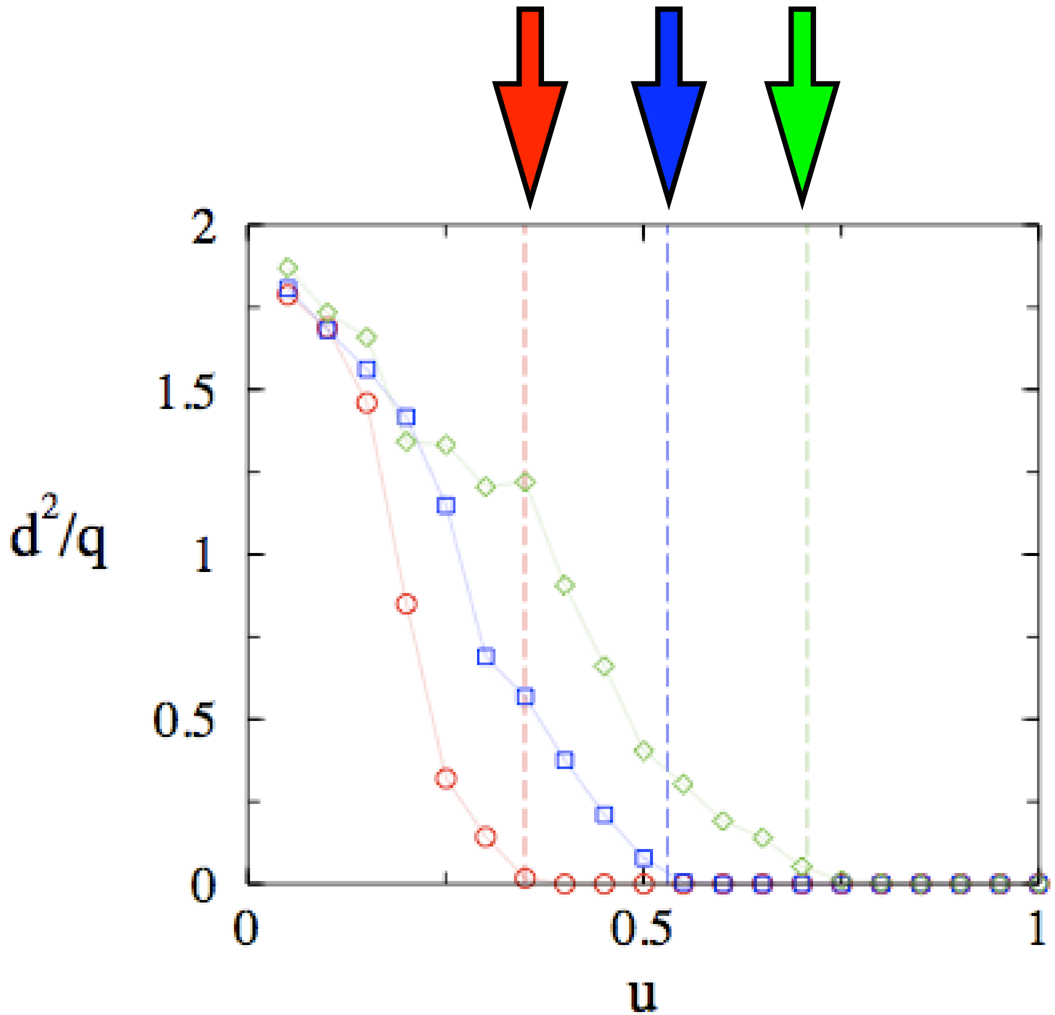


weak co-operation pressure

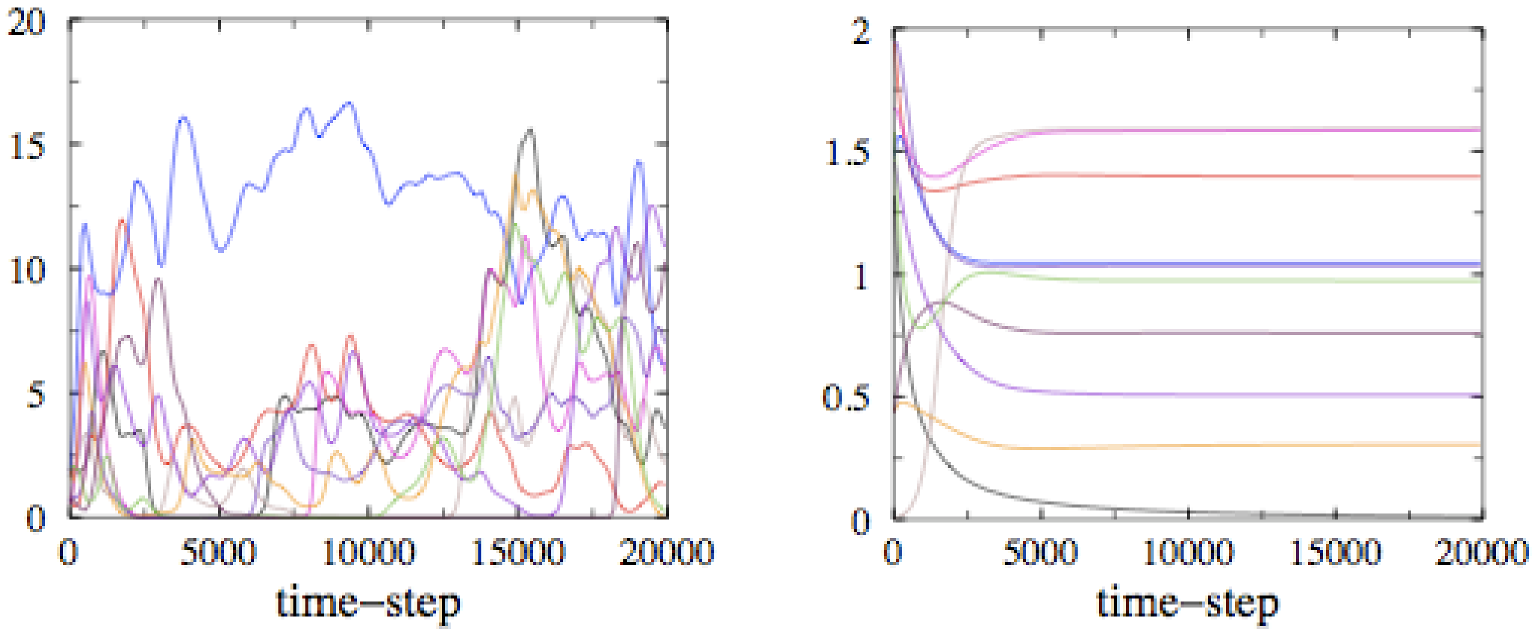
strong co-operation pressure

Ergodicity breaking - sensitivity to initial conditions

location of phase transition
from theory

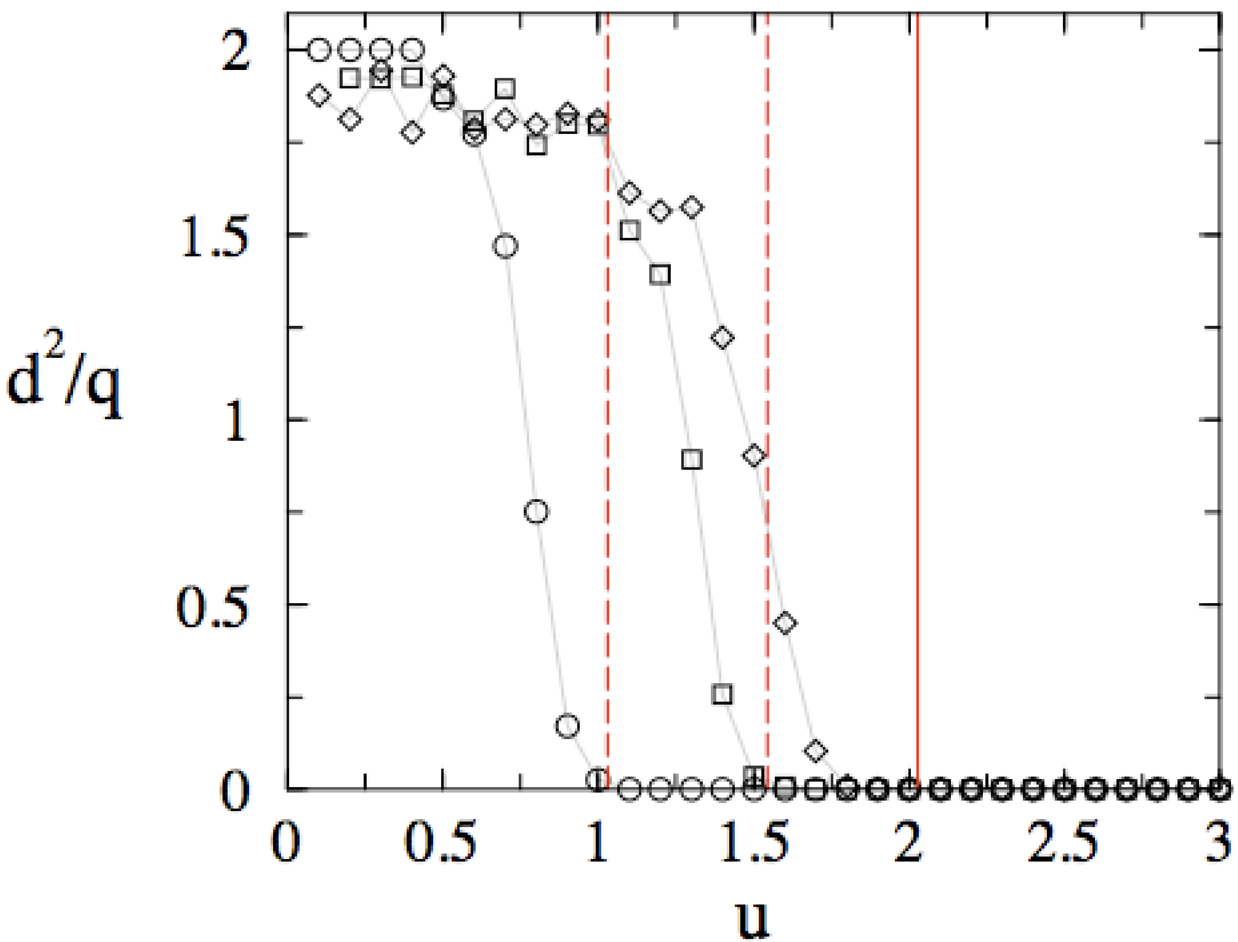


Typical trajectories



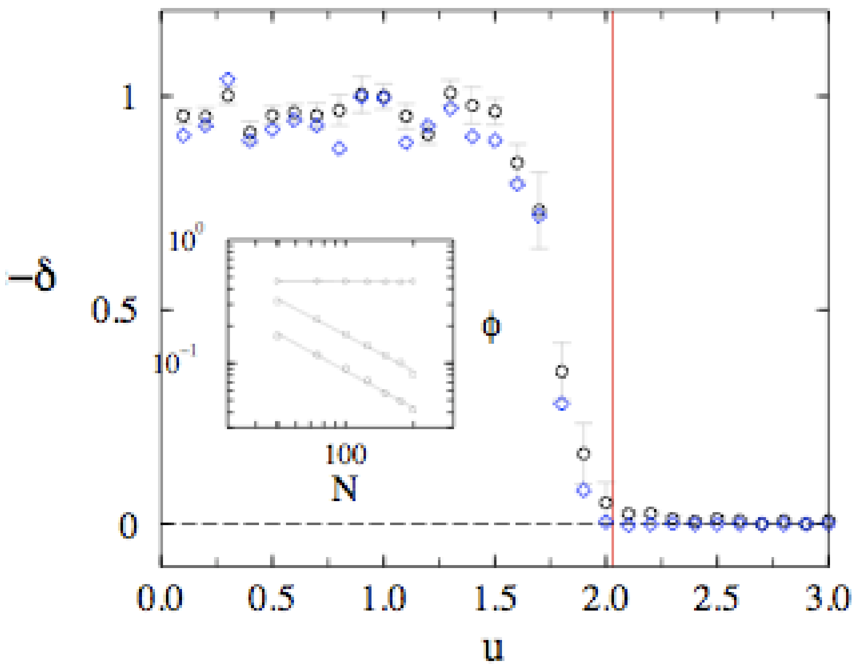
$p = 3$

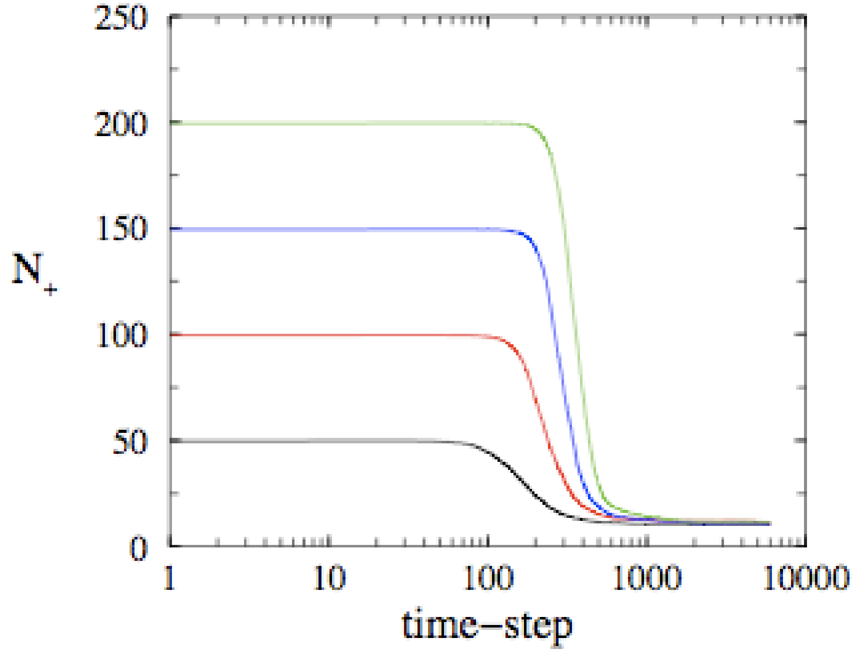
Find ergodicity breaking also for $p = 3$:



But if $p = 3$ but quadratic self-interaction find collapse of extensivity

$$\phi \sim N^{\delta}$$





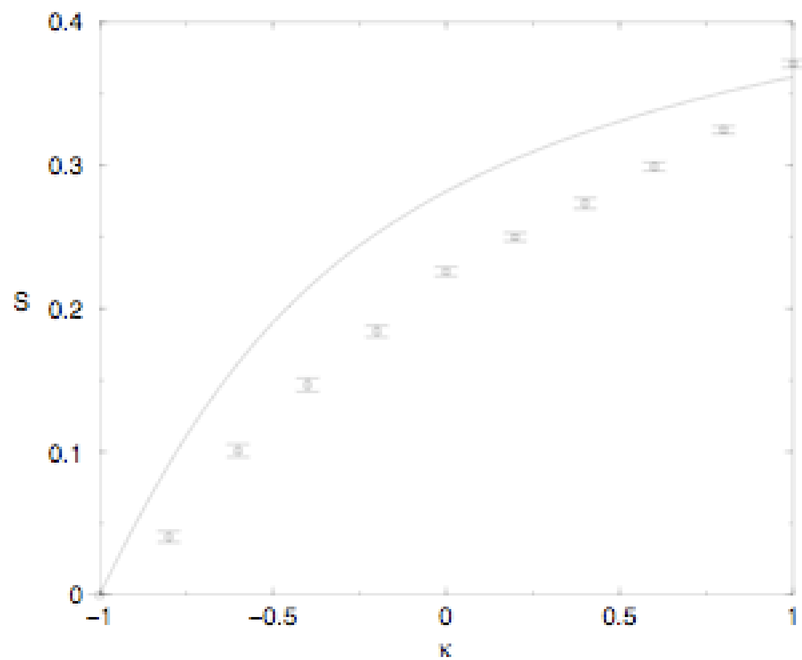
Bi-matrix games

Payoff matrices a_{ij}, b_{ij} with

$$\overline{a_{ij}^2} = \overline{b_{ij}^2} = 1/N, \qquad \overline{a_{ij}b_{ji}} = \Gamma/N$$

Berg/Weigt [Europhys. Lett. **48** 129 (1999)]:

$$S = \frac{1}{N} \ln \langle \# Nash \rangle$$



Bi-matrix games:

two-population random replicators

$$\dot{x_i} = x_i(t)\left(-2ux_i + \sum_j a_{ij}y_j - \nu^x\right)$$
$$\dot{y_j} = y_j(t)\left(-2uy_j + \sum_i b_{ij}x_i - \nu^y\right)$$

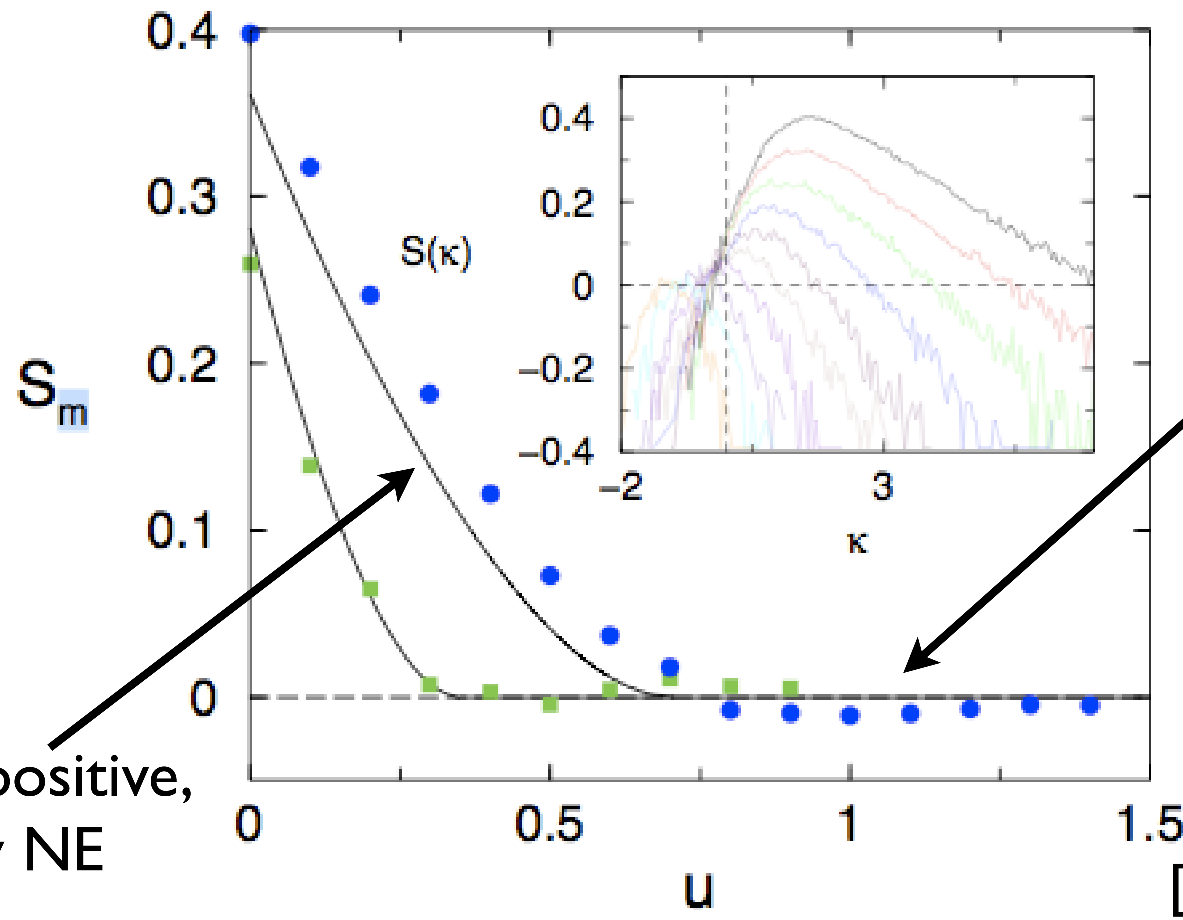
generating functionals lead to two coupled effective processes

$$\dot{x} = x(t)\left(-2ux + \Gamma\int dt' G_y(t,t')x(t') - \nu^x - \eta^x(t)\right)$$
$$\dot{y} = y(t)\left(-2uy + \Gamma\int dt' G_x(t,t')y(t') - \nu^y - \eta^y(t)\right)$$

with

$$\langle\eta^x(t)\eta^x(t')\rangle = \langle y(t)y(t')\rangle \qquad \langle\eta^y(t)\eta^y(t')\rangle = \langle x(t)x(t')\rangle$$

Dynamic instability and number of Nash equilibria



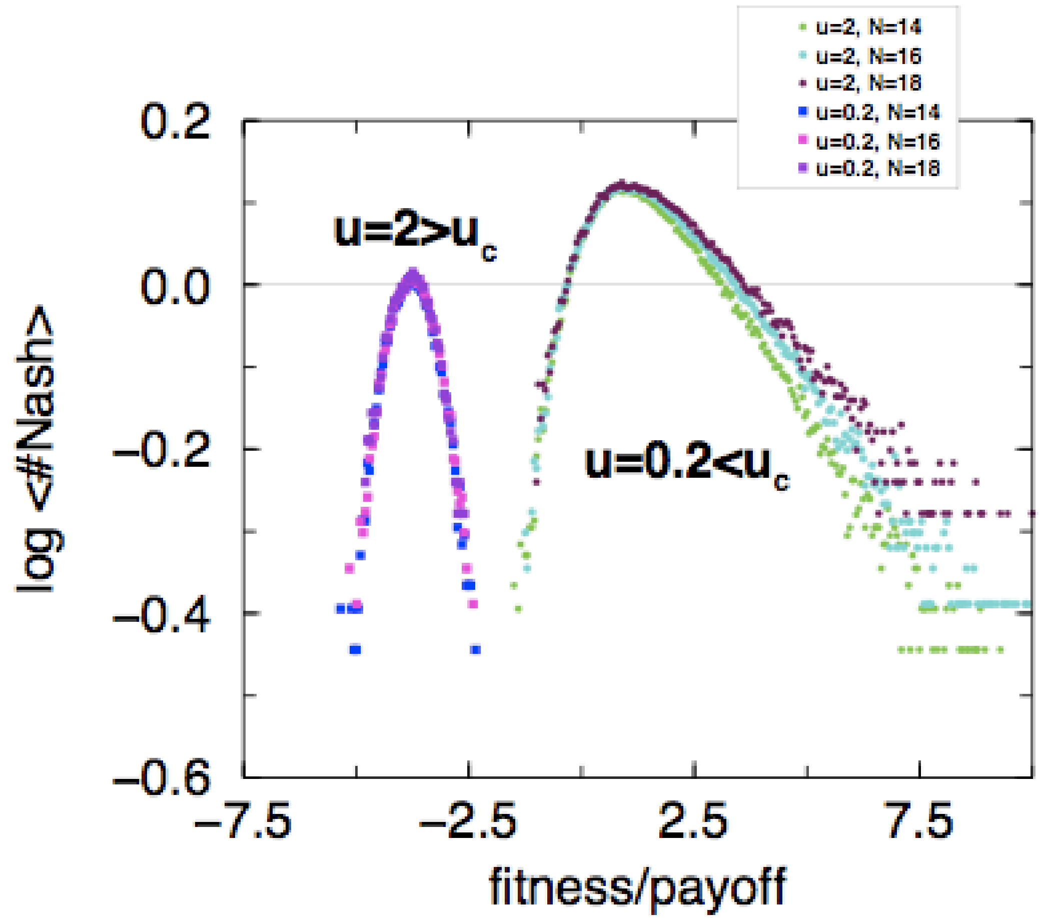
entropy zero, one NE

entropy positive, many NE

[TG, Europhys. Lett. 2007]

dynamic instability coincides with onset of exponential number of NE

Counting the Nash Equilibria



Statistical mechanics of simple model eco-systems

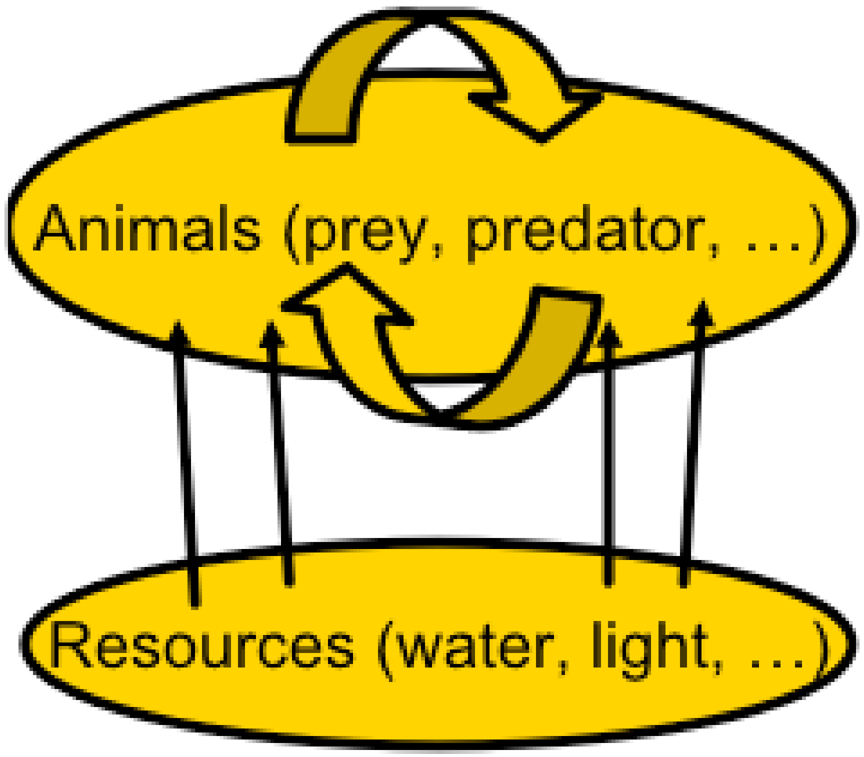
work with
Yoshimi Yoshino and Kei Tokita (Osaka)

J. Stat. Mech. (2007) P09003

Phys. Rev. E (2008) to appear

The model

two ‘trophic levels’:



Model definitions

two 'trophic levels':

- N species $i = 1, \ldots, N$
- P resources $\mu = 1, \ldots, P$

$$\alpha = \frac{P}{N}$$

fitness of species i

$$f_i = \sum_j J_{ij} x_j + \sum_\mu \xi_i^\mu A^\mu$$

direct interaction between species

use of resources

abundance of resource μ

$$A^\mu(t) = A_0^\mu - \sum_i \xi_i^\mu x_i(t)$$

Model definitions

abundance of resource μ

$$A^\mu(t) = A_0^\mu - \sum_i \xi_i^\mu x_i(t)$$

abundance in absence of species

use of resources

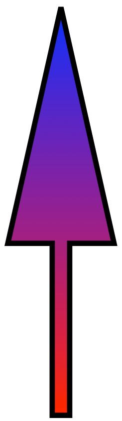
ξ_i^μ random variable of variance 1

A_0^μ random variable of variance σ^2

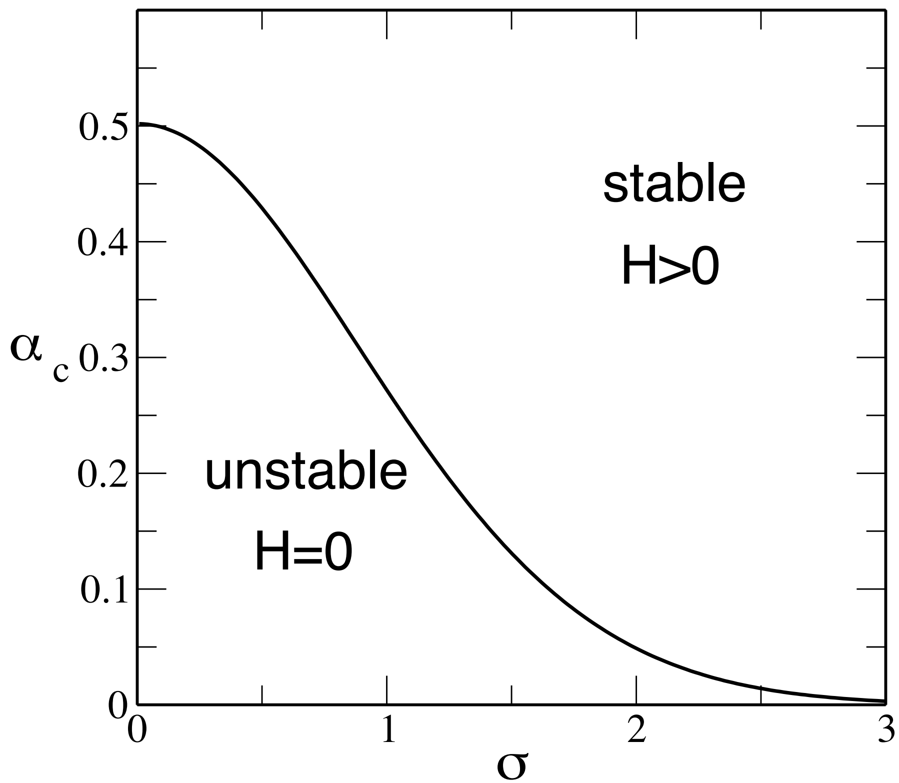
[see also A De Martino and M Marsili J. Phys. A 39 R465 (2007)]

E.g. phase diagram in dependence of number of resources and their variability

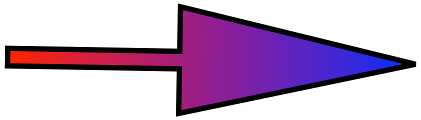
many resources



few resources



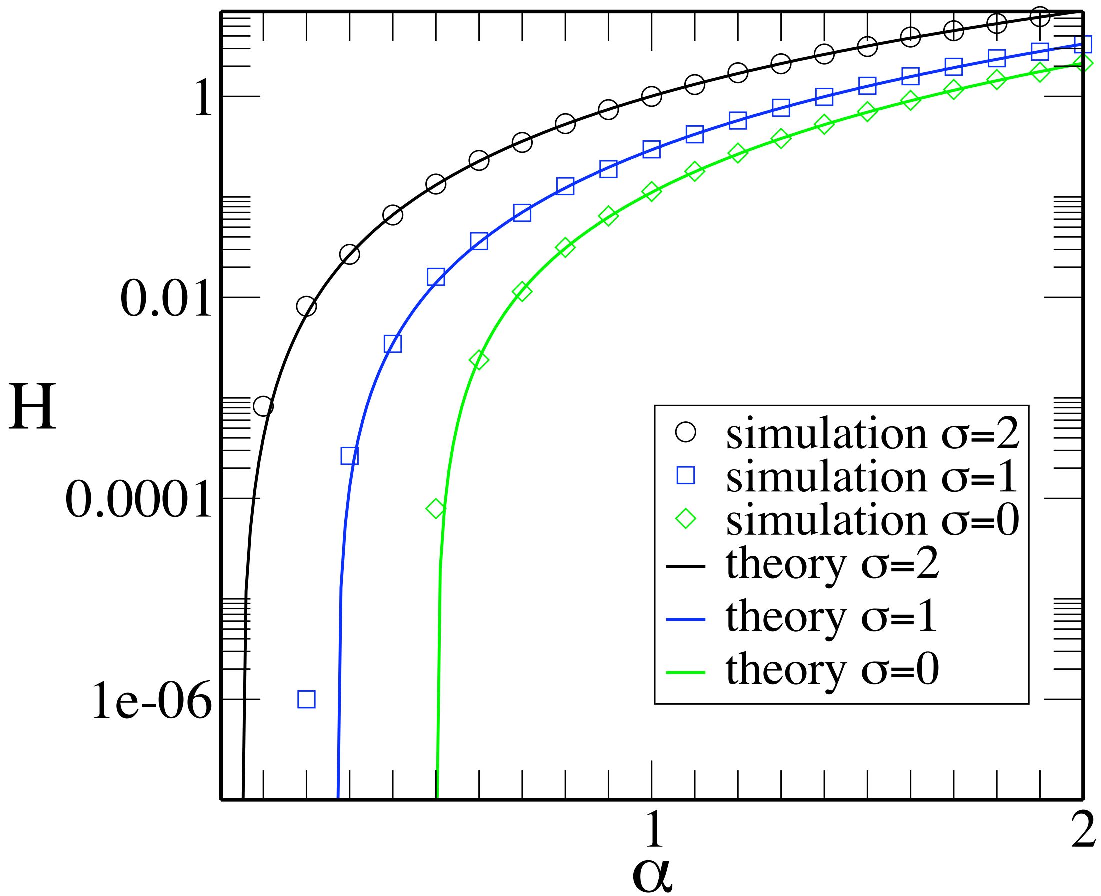
small variability



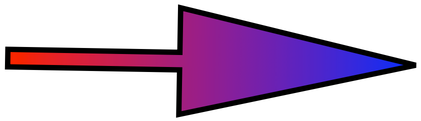
large variability

Use of resources

$$H = \frac{1}{P} \sum_{\mu=1}^{P} \langle A^{\mu}(t) \rangle^2$$

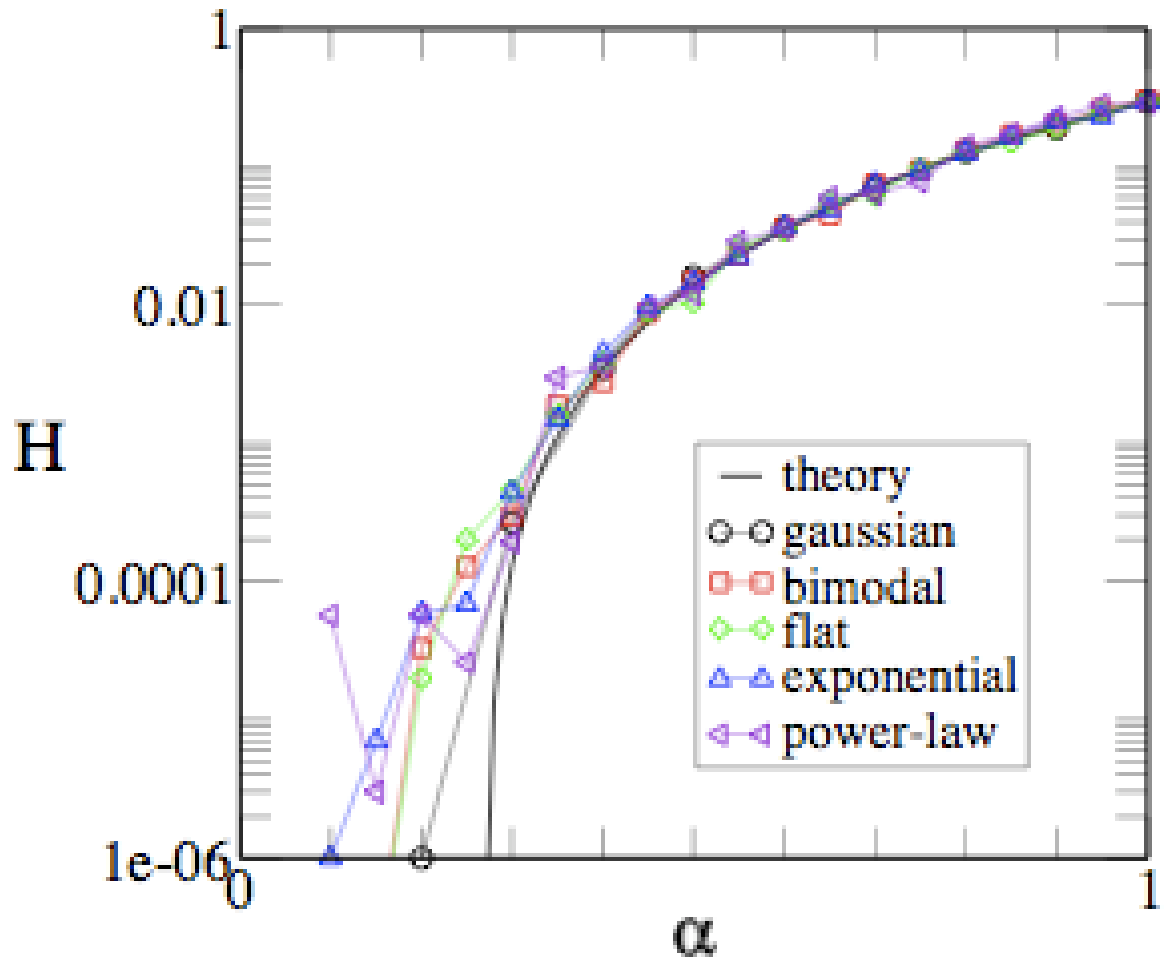


few resources

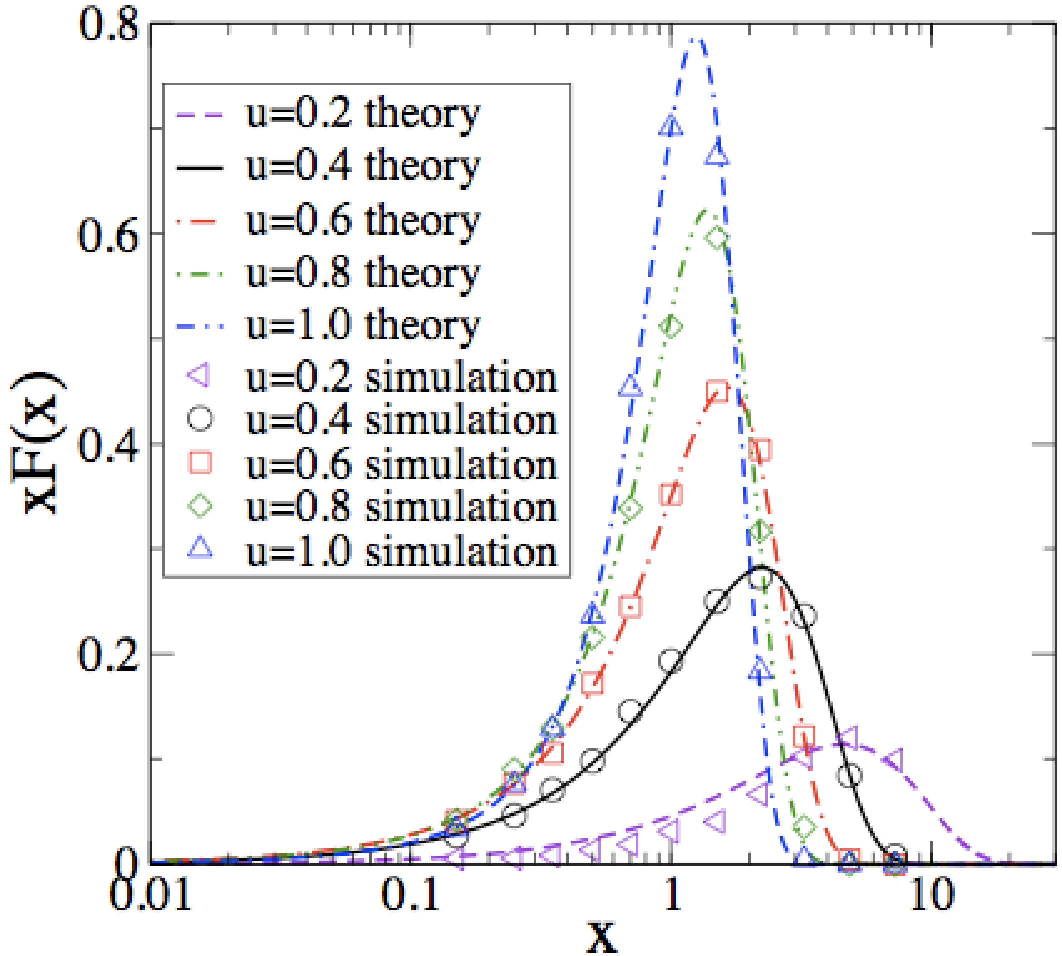


many resources

Robustness of the model:
distribution of species-resource couplings



Species abundance distributions in replicator models



[Yoshino, Galla, Tokita, PRE (2008) to appear]
[Tokita, PRL 2004]

Conclusions

- used techniques from statistical physics used to study replicator systems with random interaction matrices
- transition between ergodic-stable and non-ergodic-unstable phase
- order parameters computable in stable regime
- extension to simple model-eco system with two trophic levels
- phases with perfect exploitation of resources